$\boxed{\text{HW 8}}$

$\boxed{7.39}$ phase velocity :

$$\boxed{v_{ph} = \frac{\omega}{k}} \leftarrow \text{(is okay)}$$

(or) Using $\omega^2 = \omega_p^2 + c^2 k^2$

$$k = \sqrt{\frac{\omega^2 - \omega_p^2}{c^2}}$$

$$v_{ph} = \omega \cdot \frac{c}{\sqrt{\omega^2 - \omega_p^2}} = \boxed{\frac{c}{\sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}}} \quad \text{(also okay)}$$

$$v_{group} = \frac{d\omega}{dk} = \frac{d}{dk}\left[\sqrt{\omega_p^2 + c^2k^2}\right] = \frac{1}{2}\left(\omega_p^2 + c^2k^2\right)^{-1/2} \cdot 2c^2k$$

$$= \boxed{\frac{c^2k}{\sqrt{\omega_p^2 + c^2k^2}}} \quad \text{(is okay)}$$

$$\text{(or)} = \frac{c^2k}{ck} \frac{1}{\sqrt{\frac{\omega_p^2}{c^2k^2} + \frac{c^2k^2}{c^2k^2}}} = \frac{c}{\sqrt{1 + \frac{\omega_p^2/c^2}{\underbrace{(\omega^2-\omega_p^2)/c^2}}}} = \frac{c}{\sqrt{\frac{\omega_p^2 + \omega^2 - \omega_p^2}{\omega^2 - \omega_p^2}}}$$

[using: $\omega^2 - \omega_p^2 = c^2k^2$]

$$= c\sqrt{\frac{\omega^2 - \omega_p^2}{\omega^2}} = \boxed{c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \quad \text{(also okay!)}$$

$\boxed{7.42}$

(a) The series is neither even nor odd, so it must have cosine and sine terms.

(b) The lack of symmetry (about x-axis) requires odd & even ωt multiples.

(c) DC term $= 1/3$

(d) $A_0 = 2/3$

(e) Period $= T = \frac{2\pi}{\omega}$

($T \approx 8$ also okay)

(f) Note: $A_0 = 2/3$

$A_1 = 0$

$B_1 = 1$

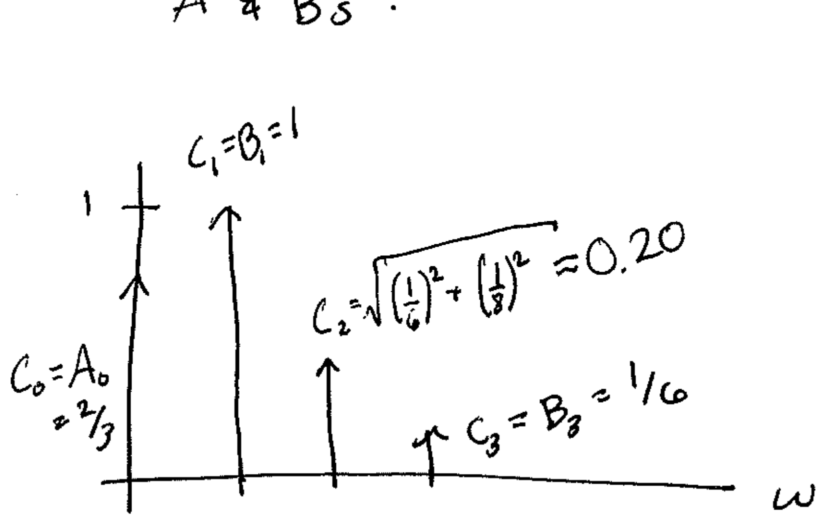
$A_2 = \frac{1}{6}$

$B_2 = \frac{1}{8}$

$A_3 = 0$

$B_3 = \frac{1}{6}$

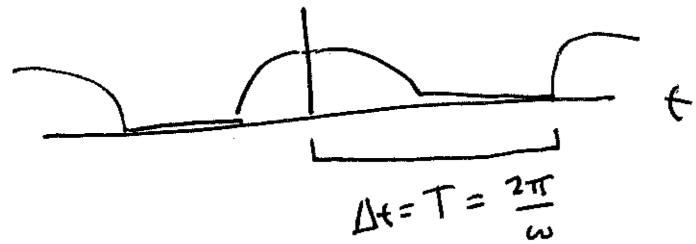
Use $C_m^2 = A_m^2 + B_m^2$ to combine A & Bs:



(Also okay to plot A & B separately)

$\boxed{7.51}$ (Extra credit)

Rectified cosine :



$$E(t) = \begin{cases} E_0 \cos \omega t , & \frac{-\pi}{2\omega} \le t \le \frac{\pi}{2\omega} \\ 0 , & \frac{\pi}{2\omega} < t < \frac{3\pi}{2\omega} \end{cases} \quad \text{and so on...}$$

· Even function, $f(-x) = f(x) \Rightarrow$ all $B_m = 0$

(1) $$A_0 = \frac{2}{T} \int_{-\pi/2\omega}^{\pi/2\omega} E_0 \cos(\omega t)\, dt$$

$$A_0 = \frac{E_0 \omega}{\pi} \int \cos \omega t \, dt = \frac{E_0 \omega}{\pi} \frac{1}{\omega} \Big[\sin \omega t \Big]_{-\pi/2\omega}^{\pi/2\omega}$$

$$= \frac{E_0}{\pi} \left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right)\right] = \boxed{\frac{2E_0}{\pi}}$$

(2) $$A_m = \frac{\omega}{\pi} E_0 \int_{-\pi/2\omega}^{\pi/2\omega} \cos(\omega t) \cos(m\omega t)\, dt$$

$$\rightarrow \cos(\omega t)\cos(m\omega t) = \frac{1}{2}\left[\cos(\omega t - m\omega t) + \cos(\omega t + m\omega t)\right]$$

$$= \frac{1}{2}\left[\cos((m-1)\omega t) + \cos((m+1)\omega t)\right]$$

$$A_m = \frac{E_0 \omega}{2\pi} \int_{-\pi/2\omega}^{\pi/2\omega} \cos\left[(m-1)\omega t\right] dt + \frac{E_0 \omega}{2\pi} \int_{\pi/2\omega}^{\pi/2\omega} \cos\left[(m+1)\omega t\right] dt$$

Substitute variables:

$$y_1 = (m-1)\omega t \qquad\qquad y_2 = (m+1)\omega t$$

$$dy_1 = (m-1)\omega\, dt \qquad\qquad dy_2 = (m+1)\omega\, dt$$

$$y_1\left(\frac{\pi}{2\omega}\right) = (m-1)\frac{\pi}{2} \qquad\qquad y_2\left(\frac{\pi}{2\omega}\right) = (m+1)\frac{\pi}{2}$$

$$y_1\left(\frac{-\pi}{2\omega}\right) = -(m-1)\frac{\pi}{2} \qquad\qquad y_2\left(\frac{-\pi}{2\omega}\right) = -(m+1)\frac{\pi}{2}$$

$$A_m = \frac{E_0 \omega}{2\pi} \int_{-(m-1)\pi/2}^{(m-1)\pi/2} \frac{\cos y_1\, dy_1}{(m-1)\,\omega} + \frac{E_0 \omega}{2\pi} \int_{-(m+1)\pi/2}^{(m+1)\pi/2} \frac{\cos y_2\, dy_2}{(m+1)\,\omega}$$

$$\frac{A_m}{E_0} = \frac{1}{2\pi}\frac{1}{(m-1)}\left[\sin y_1\right]_{-(m-1)\pi/2}^{(m-1)\pi/2} + \frac{1}{2\pi}\frac{1}{(m+1)}\left[\sin y_2\right]_{\cdots}^{\cdots}$$

$$= \frac{1}{(m-1)\,2\pi}\left[\sin\left((m-1)\frac{\pi}{2}\right) - \sin\left(-(m-1)\frac{\pi}{2}\right)\right]$$

$$+ \frac{1}{(m+1)2\pi}\left[\sin\left((m+1)\frac{\pi}{2}\right) - \sin\left(-(m+1)\frac{\pi}{2}\right)\right]$$

$$\frac{A_m}{E_0} = \frac{\sin\left((m-1)\pi/2\right)}{2\cdot(m-1)\pi/2} + \frac{\sin\left((m+1)\pi/2\right)}{2\cdot(m+1)\pi/2}$$

$$\frac{A_m}{E_0} = \frac{1}{2}\left[\operatorname{sinc}\left((m-1)\frac{\pi}{2}\right) + \operatorname{sinc}\left((m+1)\frac{\pi}{2}\right)\right]$$

($\rightarrow$ Using $\operatorname{sinc}(x) = \sin(x)/x$.)

<u>Note</u>: When m is odd, sinc is 0, for all odd $m \neq 1$.

$$A_1 = \frac{E_0 \operatorname{sinc}(0)}{2} + \frac{E_0 \sin(2\cdot\pi/2)}{2} = \frac{E_0}{2} + 0 = \boxed{\frac{1}{2}E_0}$$

$$A_2 = \frac{1}{2}\left[\operatorname{sinc}(\pi/2) + \operatorname{sinc}(3\pi/2)\right]E_0 = \boxed{\frac{2}{3\pi}} \times E_0$$

$$A_3 = \frac{1}{2}\left[\operatorname{sinc}(\pi) + \sin(2\pi)\right]E_0 = \boxed{0} \times E_0$$

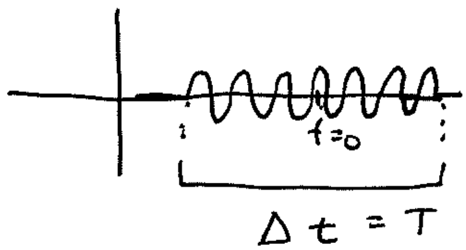
$$A_4 = \frac{1}{2}\left[\operatorname{sinc}(3\pi/2) + \operatorname{sinc}(5\pi/2)\right]E_0 = \boxed{\frac{-2}{15\pi}} \times E_0$$

$$E(t) = \frac{A_0}{2} + A_1 \cos \omega t + A_2 \cos 2\omega t + \cdots$$

Thus:

$$E(t) = E_0\left[\frac{1}{\pi} + \frac{1}{2}\cos \omega t + \frac{2}{3\pi}\cos 2\omega t + \frac{-2}{15\pi}\cos 4\omega t + \cdots\right]$$

$\boxed{7.54}$ Find $A(\omega)$ for



$E(t) = E_0 \cos \omega_p t$

within Δt

$$A(\omega) = \int_{-\infty}^{\infty} E(t) \cos(\omega t)\, dt$$

← Fourier Integral, more appropriate for a finite duration (not infinitely repeating).

$$= E_0 \int_{-T/2}^{T/2} \cos(\omega_p t) \cos(\omega t)\, dt$$

$$= E_0 \int_{\dots}^{\dots} \frac{1}{2} \cos\left((\omega_p - \omega) t\right) dt + E_0 \int_{\dots}^{\dots} \frac{1}{2} \cos\left((\omega_p + \omega) t\right) dt$$

$$= \frac{E_0}{2} 2 \left\{ \operatorname{sinc}\left[\frac{T}{2}(\omega_p - \omega)\right] + \operatorname{sinc}\left[\frac{T}{2}(\omega_p + \omega)\right] \right\} \frac{T}{2}$$

$$\boxed{A(\omega) = \frac{E_0 T}{2} \left[\operatorname{sinc}\left(\frac{T}{2}(\omega_p - \omega)\right) + \operatorname{sinc}\left(\frac{T}{2}(\omega_p + \omega)\right) \right]}$$

$$\operatorname{sinc}\left(\frac{\pi}{2}\right) = \frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi} \approx 0.64$$

← larger than $\frac{1}{2}$ for $\theta < \frac{\pi}{2}$

• At $\frac{1}{2}$ maximum (assuming this in $\approx \frac{\pi}{2}$) ...

$$\Delta\nu_{H.M.} \Rightarrow \quad -\frac{\pi}{2} \leq \frac{T}{2}(\omega_p - \omega) \leq \frac{\pi}{2}$$

$$-\frac{\pi}{T} \leq \omega_p - \omega \leq \frac{\pi}{T}$$

$$\omega_p - \frac{\pi}{T} \leq \omega \leq \omega_p + \frac{\pi}{T}$$

$$\Delta\omega_{Half.\,Max.} = 2 \cdot \frac{\pi}{T} \Rightarrow \Delta\nu = \frac{1}{T} \quad \left(= \frac{\Delta\omega}{2\pi}\right)$$

Thus, $\Delta\nu \cdot \Delta t = 1$. //

$\boxed{7.56}$ LED has $\lambda_o = 446 nm$ & $\Delta\lambda = 21 nm$

Spectrum spans $\left(\lambda_o - \frac{\Delta\lambda}{2}\right)$ to $\left(\lambda_o + \frac{\Delta\lambda}{2}\right)$

or $\lambda_{min} = 435.5 nm$ to $\lambda_{max} = 456.5 nm$

$$\Delta\nu = \nu_{max} - \nu_{min} = \frac{c}{\lambda_{min}} - \frac{c}{\lambda_{max}}$$

$$= 3\times10^{8} \, m/s \left(\frac{1}{435.5\times10^{-9} \, m} - \frac{1}{456.6\times10^{-9} \, m}\right)$$

$$= \underline{\underline{3.17 \times 10^{13} \, Hz}}$$

$$L_c = c/\Delta\nu = \frac{3\times10^{8} \, m/s}{3.17\times10^{13} \, 1/s} = \underline{\underline{9.47 \, \mu m}} = 9.47\times10^{-6} \, m$$

$$T_c = \frac{1}{\Delta\nu} = \underline{\underline{3.15 \times10^{-14} \, s}}$$

$\boxed{7.59}$

$$\frac{\Delta\nu}{\bar{\nu}} = 2 \text{ parts in } 10^{10} = \frac{2}{10^{10}} = 2 \times 10^{-10}$$

$$\bar{\nu} = \frac{c}{\bar{\lambda}} = \frac{c}{632.8\,\text{nm}} = 474.1 \text{ THz}$$

$$\Delta\nu = \bar{\nu} \cdot 2\times10^{-10} = \boxed{94.8 \text{ kHz}}$$

$$L_c = \frac{c}{\Delta\nu} = \frac{3\times10^{8}\ \text{m/s}}{9.48\times10^{4}\ 1/\text{s}} = \boxed{3.16 \text{ km}}$$

$\boxed{8.2}$ When you have 2 linear polarizations which are in-phase:

- $E = E_z + E_x$ is also linear
- $\tan\alpha = \frac{E_z}{E_x}$; $\alpha \approx 53°$ above x-axis

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$\boxed{8.3}$ You have 2 linear polarizations but E_z leads E_x by $\pi/2$:

- $E = E_z + E_x$ is right circularly polarized
- $|E_0| = 8$